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MAXIMUM PRINCIPLES FOR DEGENERATE ELLIPTIC-PARABOLIC EQUATIONS WITH VENTTSEL′'S BOUNDARY CONDITION

BY

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ABSTRACT. In this paper, we first establish interior and boundary maximum principles for degenerate elliptic-parabolic equations; we state both principles in one single theorem in terms of the *propagation set* (cf. Theorem 1). We next generalize the boundary condition to Venttsel's one and obtain the similar result (cf. Theorem 2). Venttsel′'s boundary condition contains Dirichlet, Neumann, oblique derivative and mixed boundary conditions as special cases and, from a probabilistic point of view (cf. Venttsel′ [9]), it is the most general admissible boundary condition. We give several examples in the last section.

1. Introduction. Let M be a d-dimensional C^∞ manifold with boundary ∂M. $(U, \phi = (x_1, \dots, x_d))$ denotes a chart of M, i.e., ϕ is a C^∞ diffeomorphism of an open set U on M onto an open set $\phi(U)$ on $\overline{\mathbf{R}}^d_+ = \{(x_1, \dots, x_d) \in \mathbf{R}^d\colon x_d \geqslant 0\}$. We often call $\phi = (x_1, \dots, x_d)$ coordinate system on U and identify M with $\overline{\mathbf{R}}^d_+$ on U. For a vector v in the tangent space $T_x(M)$ at x, with $x \in \partial M$, we say that v is tangential (resp. transversal) to ∂M at x if $v \in T_x(\partial M)$ (resp. $\notin T_x(\partial M)$) and further, we say that v points into (resp. out of) M at x if for some chart (U, ϕ), with $x \in U$, $(d\phi)_x v \in \mathbf{R}^d_+$ (resp. $\in \mathbf{R}^d_-$), where $\mathbf{R}^d_\pm = \{(x_1, \dots, x_d) \in \mathbf{R}^d\colon x_d \gtrless 0\}$.

By A we denote a degenerate elliptic-parabolic operator on M with real C^∞ coefficients; this means that on each chart $(U, \phi = (x_1, \dots, x_d))$, A is expressed as

$$A = \sum_{i,j=1}^{d} a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i \frac{\partial}{\partial x_i} \tag{1}$$

and

$$a = (a^{ij})_{d\times d} \in C^\infty(U, S_d),$$
$$b = (b^1, \dots, b^d) \in C^\infty(U, \mathbf{R}^d), \tag{2}$$

where S^d denotes the class of symmetric positive semidefinite $d \times d$ matrices and $C^\infty(U, S_d)$ (resp. $C^\infty(U, \mathbf{R}^d)$) is the set of all S_d (resp. $\mathbf{R}^d$)-valued C^∞ functions on U. For a chart $(U, \phi = (x_1, \dots, x_d))$ we consider vector fields

$$X_0 = \sum_{i=1}^{d} \left(b^i - \sum_{j=1}^{d} \frac{\partial a^{ij}}{\partial x_j} \right) \frac{\partial}{\partial x_i},$$

$$X_k = \sum_{j=1}^{d} a^{kj} \frac{\partial}{\partial x_j}, \qquad 1 \leqslant k \leqslant d, \tag{3}$$

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on U and define a mapping $\Delta_A|U$, which assigns a subset of $T_x(M)$ to each point $x \in U$, as follows: For each $x \in U$, $v \in (\Delta_A|U)(x)$ if and only if $v \in T_x(M)$ and

$$v = \sum_{i=0}^{d} \lambda_i X_i(x) \quad \text{for some } \lambda_0 \in \overline{\mathbf{R}}_+, \lambda_i \in \mathbf{R}, 1 \leqslant i \leqslant d,$$

where $\overline{\mathbf{R}}_+ = \{t \in \mathbf{R}: t \geqslant 0\}$. The mapping $\Delta_A|U$ is invariant relative to C^∞-smooth (nonsingular) coordinate transformations on U (cf. Lemma 4), though each of the vector fields $X_0, X_1, \ldots, X_d$ on U is not invariant. Thus we may define a mapping Δ_A on M by the relation:

$$\Delta_A = \Delta_A|U \quad \text{on every chart } (U, \phi).$$

The mapping Δ_A is not a distribution on M (in the sense of differential geometry) unless $\Delta_A = -\Delta_A$ on M.

We call a vector $v \in \Delta_A(x)$, with $x \in M$, *A-diffusion vector* (resp. *A-drift vector*) at x if $-v \in \Delta_A(x)$ (resp. $-v \notin \Delta_A(x) \setminus \{0\}$). Following Fichera [3], we classify the boundary ∂M into four parts:

$$\Sigma_3 = \{x \in \partial M: \text{At least, one of } A\text{-diffusion vectors is transversal to } \partial M \text{ at } x\},$$

$$\Sigma_2 = \{x \in \partial M \setminus \Sigma_3: \text{At least, one of } A\text{-drift vectors points out of } M \text{ at } x\},$$

$$\Sigma_1 = \{x \in \partial M \setminus \Sigma_3: \text{At least, one of } A\text{-drift vectors points into } M \text{ at } x\},$$

$$\Sigma_0 = \{x \in \partial M \setminus \Sigma_3: \text{All } A\text{-drift vectors are tangential to } \partial M \text{ at } x\}.$$

Since all A-diffusion vectors are tangential to ∂M on $\partial M \setminus \Sigma_3$, any A-drift vector never points into (resp. out of) M on Σ_2 (resp. Σ_1). Hence $\partial M = \bigcup_{i=0}^{3} \Sigma_i$ and $\Sigma_i \cap \Sigma_j = \varnothing$ if $i \neq j$.

By B we denote *Venttsel′'s differential operator* on the boundary ∂M (cf. Venttsel′ [9]) with real C^∞ coefficients; this means that for each chart $(U, \phi = (x_1, \ldots, x_d))$, with $\partial M \cap U \neq \varnothing$, B is expressed as

$$B = \sum_{i,j=1}^{d-1} \alpha^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \beta^i \frac{\partial}{\partial x_i}$$

on $\partial M \cap U$ and

$$\alpha = (\alpha^{ij})_{(d-1)\times(d-1)} \in C^\infty(\partial M \cap U, S_{d-1}),$$

$$\beta = (\beta^1, \ldots, \beta^d) \in C^\infty(\partial M \cap U, \mathbf{R}^d), \qquad \beta^d > 0, \tag{4}$$

where S_{d-1} denotes the class of symmetric positive semidefinite $(d-1) \times (d-1)$ matrices and $C^\infty(\partial M \cap U, S_{d-1})$ (resp. $C^\infty(\partial M \cap U, \mathbf{R}^d)$) is the set of all S_{d-1} (resp. $\mathbf{R}^d$)-valued C^∞ functions on $\partial M \cap U$. For a chart $(U, \phi = (x_1, \ldots, x_d))$, with $\partial M \cap U \neq \varnothing$, we set

$$Y_0 = \sum_{i=1}^{d-1} \left(\beta^i - \sum_{j=1}^{d-1} \frac{\partial \alpha^{ij}}{\partial x_j} \right) \frac{\partial}{\partial x_i} + \beta^d \frac{\partial}{\partial x_d},$$

$$Y_k = \sum_{j=1}^{d-1} \alpha^{kj} \frac{\partial}{\partial x_j}, \qquad 1 \leqslant k \leqslant d-1,$$

and define a mapping $\Delta_B|(\partial M \cap U)$, which assigns a subset of $T_x(M)$ to each point $x \in \partial M \cap U$, as follows: For each $x \in \partial M \cap U$, $v \in (\Delta_B|(\partial M \cap U))(x)$ if and only if $v \in T_x(M)$ and

$$v = \sum_{i=0}^{d-1} \mu_i Y_i(x) \quad \text{for some } \mu_0 \in \overline{\mathbf{R}}_+, \mu_i \in \mathbf{R}, 1 \leqslant i \leqslant d-1.$$

Modifying the proof of Lemma 4, we can show that the mapping $\Delta_B|(\partial M \cap U)$ is invariant relative to C^∞-smooth (nonsingular) coordinate transformations on U. Thus we may define the mapping Δ_B on ∂M by the following relation:

$$\Delta_B = (\Delta_B|(\partial M \cap U)) \ \text{ on } \partial M \cap U \ \text{ for every chart } (U, \phi) \text{ with } \partial M \cap U \neq \varnothing.$$

We call a vector $v \in \Delta_B(x)$, with $x \in \partial M$, *B-diffusion vector* (resp. *B-drift vector*) at x if $-v \in \Delta_B(x)$ (resp. $-v \notin \Delta_B(x) \setminus \{0\}$). Clearly every B-diffusion vector is tangential to the boundary ∂M and any B-drift vector never points out of M. We set

$$N = \left\{x \in \overline{\Sigma_2 \cup \Sigma_3} : \text{All } B\text{-drift vectors are tangential to } \partial M \text{ at } x\right\},$$

where $\overline{\Sigma_2 \cup \Sigma_3}$ is the closure of the set $\Sigma_2 \cup \Sigma_3$ relative to the topology of ∂M. Unless otherwise specified, we assume that

> N is a $(d-1)$-dimensional regular C^∞ submanifold of ∂M, with boundary ∂N.

The case $\dim N \leqslant d-2$ will be discussed in §5. We consider the following sets on the boundary ∂N:

$$\Gamma_3 = \{x \in \partial N: \text{At least, one of } B\text{-diffusion vectors is transversal to } \partial N \text{ at } x\},$$
$$\Gamma_2 = \{x \in \partial N \setminus \Gamma_3: \text{At least, one of } B\text{-drift vectors points out of } N \text{ at } x\},$$
$$\Gamma_1 = \{x \in \partial N \setminus \Gamma_3: \text{At least, one of } B\text{-drift vectors points into } N \text{ at } x\},$$
$$\Gamma_0 = \{x \in \partial N \setminus \Gamma_3: \text{All } B\text{-drift vectors are tangential to } \partial N \text{ at } x\}.$$

As in the classification of ∂M, we can show that $\partial N = \bigcup_{i=0}^{3} \Gamma_i$ and $\Gamma_i \cap \Gamma_j = \varnothing$ if $i \neq j$.

A *propagation vector field* X is a vector field on M such that

$$X \text{ never points out of } M \text{ on } \partial M \tag{5}$$

(i.e., at each point on ∂M, either X is tangential to ∂M or points into M) and such that

$$X \in \Delta_A \quad \text{on } (M \setminus N) \cup \overline{\Gamma_2 \cup \Gamma_3}, \qquad X \in \Delta_B \quad \text{on } N,$$

where $\overline{\Gamma_2 \cup \Gamma_3}$ denotes the closure of the set $\Gamma_2 \cup \Gamma_3$ relative to the topology of ∂N. By $\mathfrak{X}$ we denote the set of all propagation vector fields. We define a subset $\mathfrak{X}_1$ (resp. $\mathfrak{X}_2$) of $\mathfrak{X}$ as a set of all propagation vector fields X satisfying the following conditions:

$$-X(x) \in \Delta_A(x) \quad \text{if } x \in \overline{\Gamma_2 \cup \Gamma_3} \text{ and } X(x) \in T_x(\partial N)$$
$$\left(\text{resp. } -X(x) \in \Delta_A(x) \cap \Delta_B(x) \text{ if } x \in \partial\Sigma_3 \cup \overline{\Gamma_2 \cup \Gamma_3}\right),$$

where $\partial\Sigma_3$ is the boundary of the set Σ_3 relative to the topology of ∂M. An *$\mathfrak{X}_k$-propagation vector field* X, $k = 1, 2$, is a propagation vector field X which belongs to the class $\mathfrak{X}_k$. We call a curve $x(t)$, $0 \leqslant t \leqslant T$ $(0 < T \leqslant \infty)$ on the manifold M *$\mathfrak{X}_k$-propagation path*, $k = 1, 2$, if there is a vector field $X \in \mathfrak{X}_k$ such that

$$\dot{x}(t) = X(x(t)) \quad \text{on } [0, T), \tag{6}$$

$$\begin{array}{l} X \text{ is a } C^1 \text{ vector field on a neighborhood of } \{x(t) \in (M \setminus N) \\ \cup \overline{\Gamma_2 \cup \Gamma_3}\colon 0 \leqslant t < T\} \text{ on } M \end{array} \tag{7}$$

and

$$\begin{array}{l} X \text{ is a } C^1 \text{ vector field on a neighborhood of } \{x(t) \in N\colon \\ 0 \leqslant t < T\} \text{ on } N. \end{array} \tag{8}$$

An $\mathfrak{X}_k$-propagation path is oriented by the parameter t. In fact, $x(-t)$ may fail to be an $\mathfrak{X}_k$-propagation path for some $\mathfrak{X}_k$-propagation path $x(t)$. For a point $p \in M$ we define the *$\mathfrak{X}_k$-propagation set* $S_k(p)$, $k = 1, 2$, as a set of all points $q \in M$ such that p and q can be connected by a finite chain of $\mathfrak{X}_k$-propagation paths running from p to q, i.e., $q \in S_k(p)$ if and only if there is a finite sequence of $\mathfrak{X}_k$-propagation paths $\{x^i(t), 0 \leqslant t < T_i\}_{i=0}^l$ satisfying $x^0(0) = p$, $\lim_{t \uparrow T_i} x^i(t) = x^{i+1}(0)$, $0 \leqslant i \leqslant l - 1$, and $\lim_{t \uparrow T_l} x^l(t) = q$. By $\bar{S}_k(p)$ we denote the closure of the propagation set $S_k(p)$ relative to the topology of M. Since $\mathfrak{X}_k$-propagation paths depend continuously on their initial conditions, $\bar{S}_k(q) \subset \bar{S}_k(p)$ if $q \in \bar{S}_k(p)$; thus the set $\bar{S}_k(p)$ is maximal in a sense.

THEOREM 1. *Assume that B is of first order and u is a solution of the oblique derivative problem*

$$Au \geqslant 0 \quad \text{on } M, \qquad Bu \geqslant 0 \quad \text{on } \Sigma_2 \cup \Sigma_3 \tag{9}$$

in $C^2(M)$. If u attains the maximum at a point p on M, then $u \equiv u(p)$ on $\bar{S}_1(p)$.

Standard arguments in maximum principle will show the following.

THEOREM 1′. *Assume that B is of first order and u is a solution of the mixed boundary value problem*

$$(A - c)u \geqslant 0 \quad \text{on } M,$$
$$(B - \gamma - \delta(A - c))u \geqslant 0 \quad \text{on } \Sigma_2 \cup \Sigma_3$$

in $C^2(M)$, where c is a nonnegative function on M and γ, δ are both nonnegative functions on ∂M. If u attains the positive maximum at a point p on M, then $u \equiv u(p)$ on $\bar{S}_1(p)$.

Interior and boundary maximum principles for degenerate elliptic-parabolic equations had been stated in various manners (cf. Bony **[2]**, Hill **[4]**, Redheffer **[7]**, Strook-Varadhan **[8]** and Myers **[5]**). Theorems 1 and 1′ contain interior principles of Bony, Hill and Redheffer as special cases and they are also generalizations of boundary principles of Myers. Amano **[1]** gave a uniform representation of both principles as stated in Theorems 1 and 1′ on more restrictive conditions as $(\Sigma_0 \cup \Sigma_1 \cup \Sigma_2) \cap \bar{\Sigma}_3 = \varnothing$, $N = \varnothing$, etc. In the proof of Theorem 1, when the point $p \in \partial\Sigma_3 \cup \partial N$ several difficulties occur (see Steps 3 and 4 of the proof of Theorem 1). However, some technical arguments (see Lemma 5, Propositions 3 and 5) enable us to remove the restrictive conditions.

We can, also, prove the similar results as Theorems 1 and 1′ when B is of second order.

THEOREM 2. *Assume that u is a solution of Venttsel″s boundary value problem*

$$Au \geqslant 0 \quad \text{on } M, \qquad Bu \geqslant 0 \quad \text{on } \Sigma_2 \cup \Sigma_3 \tag{10}$$

in $C^2(M)$. If u attains the maximum at a point p on M, then $u \equiv u(p)$ on $\bar{S}_2(p)$.

THEOREM 2′. *Assume that u is a solution of Venttsel″s boundary value problem*

$$(A - c)u \geqslant 0 \quad \text{on } M,$$
$$(B - \gamma - \delta(A - c))u \geqslant 0 \quad \text{on } \Sigma_2 \cup \Sigma_3$$

in $C^2(M)$, where the functions c, γ and δ are taken as in Theorem 1′. If u attains the positive maximum at a point p on M, then $u \equiv u(p)$ on $\bar{S}_2(p)$.

Theorem 1 is not a special case of Theorem 2 since $\bar{S}_1(p)$ is not always contained in $\bar{S}_2(p)$.

As is well known, the maximum principle provides a routine method for proving the uniqueness of classical solutions. The following theorems will illustrate how the global property of the diffusion and drift vectors may be connected with the uniqueness question.

THEOREM 3. *Assume that M is compact, B is of first order and u is a solution of the oblique derivative problem*

$$Au = 0 \quad \text{on } M, \qquad Bu = 0 \quad \text{on } \Sigma_2 \cup \Sigma_3$$

in $C^2(M)$. If

$$\bar{S}_1(p) \cap \bar{S}_1(q) \neq \varnothing \quad \text{for every } p, q \in M$$

then $u \equiv$ constant on M.

THEOREM 4. *Assume that M is compact and u is a solution of Venttsel″s boundary value problem*

$$(A - c)u = 0 \quad \text{on } M,$$
$$(B - \gamma)u = 0 \quad \text{on } \Sigma_2 \cup \Sigma_3$$

in $C^2(M)$, where c and γ are respectively nonnegative functions defined on M and ∂M. If

$$\bar{S}_2(p) \cap \bar{\Sigma}_D \neq \varnothing \quad \text{for every } p \in M,$$

then $u \equiv 0$ on M, where $\Sigma_D = \{x \in \Sigma_2 \cup \Sigma_3\colon \Delta_B(x) = \{0\}$ and $\gamma(x) \neq 0\}$ (this implies $u = 0$ on Σ_D) and $\bar{\Sigma}_D$ is the closure of Σ_D relative to the topology of ∂M.

We required C^∞-smoothness in (2) and (4) only for simplicity. It will suffice to assume the following instead: M and N are C^3 manifolds with boundaries,

$$(a^{ij})_{d\times d} \in C^1(U, S_d),$$
$$(b^1, \dots, b^d) \in C^0(U, \mathbf{R}^d) \tag{2′}$$

and

$$(\alpha^{ij})_{(d-1)\times(d-1)} \in C^1(\partial M \cap U, S_{d-1}),$$
$$(\beta^1, \ldots, \beta^d) \in C^2(\partial M \cap U, \mathbf{R}^d), \qquad \beta^d > 0. \tag{4'}$$

We can also replace the C^1-smoothness conditions in (7) and (8) by local Lipschitz continuity conditions.

2. Preliminaries. In this section we shall give six basic lemmas.

LEMMA 1 ([1]). *Assume that $a(t) \in C^1(\mathbf{R}, S_d)$. Then for any $\lambda, \mu \in \mathbf{R}^d$*
(i) $\langle a(0)\lambda, \lambda\rangle = 0$ *implies* $a(0)\lambda = 0$,
(ii) $\langle a(0)\lambda, \lambda\rangle = 0$, $\langle a(0)\mu, \mu\rangle = 0$ *imply* $\langle \dot{a}(0)\lambda, \mu\rangle = 0$.

Here $\langle\ ,\ \rangle$ denotes the inner product in $\mathbf{R}^d$. For $(a^{ij}(t))_{d\times d} \in C^1(\mathbf{R}, S_d)$ Lemma 1 shows that if $a^{kk}(0) = 0$ then $a^{ik}(0) = a^{ki}(0) = 0$, $1 \leqslant i \leqslant d$, and that if $a^{kk}(0) = a^{ll}(0) = 0$ then $\dot{a}^{kl}(0) = \dot{a}^{lk}(0) = 0$.

LEMMA 2 ([1]). *Assume that $X(x) = \sum_{i=1}^d a^i(x)\partial/\partial x_i$ is a C^1 vector field on $\mathbf{R}^d$, with $X(0) = \partial/\partial x_1$. Let $x(t, y) = (x_1(t, y), \ldots, x_d(t, y))$ be a solution of the initial value problem $\dot{x} = X(x)$, $x(0, y) = y$. Then*

$$x_i(t, y) = y_i + \delta_{i1}t + \frac{1}{2}\frac{\partial a^i}{\partial x_1}(0)t^2 + \sum_{j=1}^d \frac{\partial a^i}{\partial x_j}(0)y_j t + o(|t|^2 + |y|^2)$$

as $|t| + |y| \to 0$, where $\delta_{i1} = 1$ and $= 0$ if $i \neq 1$.

LEMMA 3 ([1]). *Assume that F is a closed set in $\mathbf{R}^d$ and $X(x)$ is a locally Lipschitz continuous vector field on $\mathbf{R}^d$. Let $x(t)$ be a solution of the initial value problem $\dot{x} = X(x)$, $x(0) \in F$. Then $\{x(t)\colon 0 \leqslant t \leqslant T\} \subset F$ if and only if the following condition is satisfied: For any $t \in [0, T]$*

$$\langle X(y), x(t) - y\rangle \leqslant 0$$

whenever $y \in F$ and $|x(t) - y| = \operatorname{dist}(x(t), F)$.

Let us take two charts $(U, (x_1, \ldots, x_d))$ and $(U, (\bar{x}_1, \ldots, \bar{x}_d))$ of the manifold M and assume that on $(U, (x_1, \ldots, x_d))$ the differential operator A is expressed as (1) and on $(U, (\bar{x}_1, \ldots, \bar{x}_d))$ expressed as

$$A = \sum_{i,j=1}^d \bar{a}^{ij}\frac{\partial^2}{\partial \bar{x}_i \partial \bar{x}_j} + \sum_{i=1}^d \bar{b}^i \frac{\partial}{\partial \bar{x}_i}.$$

On the chart $(U, (x_1, \ldots, x_d))$ we consider vector fields $X_0, X_1, \ldots, X_d$ defined by (3) and on the chart $(U, (\bar{x}_1, \ldots, \bar{x}_d))$ consider vector fields $\bar{X}_0, \bar{X}_1, \ldots, \bar{X}_d$ defined by

$$\bar{X}_0 = \sum_{i=1}^d \left(\bar{b}^i - \sum_{j=1}^d \frac{\partial \bar{a}^{ij}}{\partial \bar{x}_j}\right)\frac{\partial}{\partial \bar{x}_i},$$
$$\bar{X}_k = \sum_{j=1}^d \bar{a}^{kj}\frac{\partial}{\partial \bar{x}_j}, \qquad 0 \leqslant k \leqslant d.$$

We denote the transition function: $(x_1, \ldots, x_d) \to (\bar{x}_1, \ldots, \bar{x}_d)$ by ψ.

Then we have the following lemma.

LEMMA 4 [1]. *For any real-valued functions* $\lambda_0, \lambda_1, \dots, \lambda_d$ *on* U *there are real-valued functions* $\bar{\lambda}_0, \bar{\lambda}_1, \dots, \bar{\lambda}_d$ *on* U *such that*

$$d\psi\left(\sum_{i=0}^{d} \lambda_i X_i\right) = \sum_{i=0}^{d} \bar{\lambda}_i \bar{X}_i \quad and \quad \lambda_0 = \bar{\lambda}_0.$$

LEMMA 5. *For any point* $p \in \overline{\Sigma_2 \cup \Sigma_3} \setminus N$ *there exists a chart* $(U, (x_1, \dots, x_d))$, *with* $p \in U$, *for which Venttsel'’s differential operator* B *is expressed as*

$$B = \sum_{i,j=1}^{d-1} \alpha^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \beta^d \frac{\partial}{\partial x_d}, \qquad \beta^d > 0,$$

on $\partial M \cap U = \{(x_1, \dots, x_d) \in U: x_d = 0\}$.

PROOF. Let us choose a sufficiently small open neighborhood U of the point p on M and take a chart $(U, (\bar{x}_1, \dots, \bar{x}_d))$ arbitrarily. B is expressed as

$$B = \sum_{i,j=1}^{d-1} \bar{\alpha}^{ij} \frac{\partial^2}{\partial \bar{x}_i \partial \bar{x}_j} + \sum_{i=1}^{d} \bar{\beta}^d \frac{\partial}{\partial \bar{x}_i}, \qquad \bar{\beta}^d \geqslant 0,$$

on $\partial M \cap U = \{(\bar{x}_1, \dots, \bar{x}_d) \in U: \bar{x}_d = 0\}$. By (4) we may extend the coefficients $\bar{\alpha}^{ij}, \bar{\beta}^i$, $1 \leqslant i, j \leqslant d$, so as to be C^∞-smooth in a neighborhood of the point p in $\mathbf{R}^d$. We define functions $x_k = x_k(\bar{x}_1, \dots, \bar{x}_d)$, $1 \leqslant k \leqslant d-1$, as solutions of the initial value problems

$$\sum_{i=1}^{d} \bar{\beta}^i \frac{\partial x_k}{\partial \bar{x}_i} = 0, \qquad x_k(\bar{x}_1, \dots, \bar{x}_{d-1}, 0) = \bar{x}_k.$$

Since $p \notin N$ implies $\bar{\beta}^d(p) > 0$, each of the above initial value problems possesses a unique solution. In the new coordinate system

$$\begin{aligned} x_1 &= x_1(\bar{x}_1, \dots, \bar{x}_d), \\ &\vdots \\ x_{d-1} &= x_{d-1}(\bar{x}_1, \dots, \bar{x}_d), \\ x_d &= \bar{x}_d \end{aligned}$$

the differential operator B is rewritten as

$$B = \sum_{i,j=1}^{d-1} \alpha^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \beta^i \frac{\partial}{\partial x_i},$$

on $\partial M \cap U$, where

$$\alpha^{ij} = \sum_{k,l=1}^{d-1} \bar{\alpha}^{kl} \frac{\partial x_i}{\partial \bar{x}_k} \frac{\partial x_j}{\partial \bar{x}_l},$$

$$\beta^i = \sum_{k,l=1}^{d-1} \bar{\alpha}^{kl} \frac{\partial^2 x_i}{\partial \bar{x}_k \partial \bar{x}_l} + \sum_{k=1}^{d} \bar{\beta}^k \frac{\partial x_i}{\partial \bar{x}_k}.$$

Hence we have $\alpha^{ij} = \bar{\alpha}^{ij}$, $\beta^i = 0$, $1 \leqslant i, j \leqslant d-1$, $\beta^d = \bar{\beta}^d > 0$ on $\partial M \cap U = \{(\bar{x}_1, \dots, \bar{x}_d) \in U: \bar{x}_d = 0\}$.

LEMMA 6. *Assume that u is a solution of Venttsel''s boundary value problem* (10) *in* $C^2(M)$ *and that u attains the maximum at a point p on M. If there is a function $f \in C^2(M)$ such that $f(p) = 0$, $\nabla f(p) \neq 0$ and if one of the following conditions is satisfied*:

(i) $p \in M \setminus \overline{\Sigma_2 \cup \Sigma_3}$, $Af(p) > 0$,

(ii) $p \in \overline{\Sigma_2 \cup \Sigma_3} \setminus N$, $Af(p) > 0$, $Bf \geqslant 0$, *on a neighborhood of p on ∂M*,

(iii) $p \in N \setminus \overline{\Gamma_2 \cup \Gamma_3}$, $Bf(p) > 0$,

(iv) $p \in \overline{\Gamma_2 \cup \Gamma_3}$, $Af(p) > 0$, $Bf(p) > 0$,

then there exists a sequence $\{p^n\}_{n=1}^{\infty}$ such that $p^n \to p$ as $n \to \infty$, $p^n \in \{x \in M: f(x) \geqslant 0\} \setminus \{p\}$ and $u(p^n) = u(p)$ for $n = 1, 2, \ldots$.

From Lemma 6 a generalized form of boundary maximum principle follows. In fact, we can show the following fact: Assume that $u \in C^2(M)$ attains the maximum at a point p on ∂M but does not attain the maximum on $\mathring{M} = M \setminus \partial M$ and that $Au \geqslant 0$ on M. Then we have $Bu(p) < 0$ for Venttsel''s differential operator B if there is a function $f \in C^2(M)$ such that $f(p) = 0$, $\nabla f(p) \neq 0$, $Af(p) > 0$, $Bf(p) > 0$ and $\{x \in M: f(x) \geqslant 0\} \setminus \{p\} \subset \mathring{M}$.

PROOF OF LEMMA 6. We assume that there is a neighborhood U of the point p on M such that

$$u < u(p) \quad \text{on } \{x \in U: f(x) \geqslant 0\} \setminus \{p\}.$$

Let us choose a sufficiently small open neighborhood V of p on M, with $\overline{V} \subset U$, and take a positive number ε so that

$$0 < \varepsilon < \frac{u(p) - \sup\{u(x): x \in \overline{V} \setminus V, f(x) > 0\}}{\sup\{f(x): x \in \overline{V} \setminus V, f(x) > 0\}}.$$

Then we easily have the inequalities

$$\begin{aligned} A(u + \varepsilon f) &> 0 && \text{on } \overline{V}, \\ (u + \varepsilon f) &< u(p) && \text{on } \overline{V} \setminus V. \end{aligned} \tag{11}$$

Case (i). In case $p \in \mathring{M} = M \setminus \partial M$ we may assume that $\overline{V} \cap \partial M = \varnothing$. The former inequality in (11) implies that the function $(u + \varepsilon f)|\overline{V}$ cannot attain the maximum on V. Thus, by the latter inequality in (11), we have a contradiction $u(p) = (u + \varepsilon f)(p) < u(p)$. In case $p \in \partial M \setminus \overline{\Sigma_2 \cup \Sigma_3}$ we fix a chart $(V, (x_1, \ldots, x_d))$. We consider a differential operator

$$\tilde{A} = \sum_{i,j=1}^{d-1} a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d-1} b^i \frac{\partial}{\partial x_i} \tag{12}$$

which is a formal projection of the differential operator A, expressed as (1), upon the boundary $\partial M \cap V = \{x \in V: x_d = 0\}$. The inequalities (11) require that the function $(u + \varepsilon f)|\overline{V}$ should attain the maximum at some point, say q, on $\partial M \cap V$; this easily gives $\tilde{A}(u + \varepsilon f)(q) \leqslant 0$. If we have taken V so small that $V \subset \partial M \setminus \overline{\Sigma_2 \cup \Sigma_3}$, then we obtain

$$a^{id}(q) = a^{di}(q) = 0, \qquad \frac{\partial a^{di}}{\partial x_k}(q) = \frac{\partial a^{di}}{\partial x_k}(q) = 0$$

for $1 \leqslant i \leqslant d$, $1 \leqslant k \leqslant d-1$ by Lemma 1, and

$$\frac{\partial a^{dd}}{\partial x_d}(q) \geqslant 0, \qquad \langle X_0, e_d\rangle(q) \geqslant 0,$$

where $e_d = (0, \ldots, 0, 1) \in \mathbf{R}^d$. Hence

$$\begin{aligned}\tilde{A}(u+\varepsilon f)(q) &= A(u+\varepsilon f)(q) - b^d(q)\frac{\partial(u+\varepsilon f)}{\partial x_d}(q)\\ &= A(u+\varepsilon f)(q) - \left\{\frac{\partial a^{dd}}{\partial x_d}(q) + \langle X_0, e_d\rangle(q)\right\}\frac{\partial(u+\varepsilon f)}{\partial x_d}(q)\\ &\geqslant A(u+\varepsilon f)(q) > 0; \end{aligned} \tag{13}$$

this is a contradiction.

Case (ii). We choose V so small that $N \cap V = \varnothing$ and fix a chart $(V, (x_1, \ldots, x_d))$. Then B is expressed as

$$B = \sum_{i,j=1}^{d-1} \alpha^{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \beta^i \frac{\partial}{\partial x_i}, \qquad \beta^d > 0,$$

on $\partial M \cap V = \{(x_1, \ldots, x_d) \in V: x_d = 0\}$. The inequalities (11) show that the function $(u+\varepsilon f)|\bar{V}$ attains the maximum at some point q on $\partial M \cap V$ but cannot attain the maximum on $\mathring{M} \cap V$. In case $q \in \partial M \setminus \overline{\Sigma_2 \cup \Sigma_3}$ the same computation as (13) leads us to a contradiction. Now we note that if $q \in \overline{\Sigma_2 \cup \Sigma_3}$ then

$$0 \leqslant B(u+\varepsilon f)(q) \leqslant \beta^d(q)\frac{\partial(u+\varepsilon f)}{\partial x_d}(q) \leqslant 0$$

and $\beta^d > 0$ imply

$$\frac{\partial(u+\varepsilon f)}{\partial x_d}(q) = 0. \tag{14}$$

In case $q \in \Sigma_3$ the boundary maximum principle (cf. Hill [4]) shows $\partial(u+\varepsilon f)(q)/\partial x_d < 0$, and in case $q \in \overline{\Sigma_2 \cup \Sigma_3} \setminus \Sigma_3$ we obtain, by Lemma 1 and (14),

$$0 < A(u+\varepsilon f)(q) = \tilde{A}(u+\varepsilon f)(q) \leqslant 0;$$

this is a contradiction.

Case (iii). In this case we only have to repeat the same arguments as in Case (i).

Case (iv). If the function $(u+\varepsilon f)|\bar{V}$ attains the maximum at some point $q \in \overline{\Sigma_2 \cup \Sigma_3}$, then we have

$$0 < B(u+\varepsilon f)(q) \leqslant 0,$$

and if $q \in M \setminus \overline{\Sigma_2 \cup \Sigma_3}$ then as in Case (i) we obtain a contradiction. The proof of Lemma 6 is now complete.

3. Proof of Theorem 1. Throughout this section u denotes a solution of the oblique derivative problem (9) in $C^2(M)$, and we assume that u attains the maximum at a point p on M. For convenience sake we fix a chart $(U, (x_1, \ldots, x_d))$, with $p \in U$, i.e., we identify M with $\overline{\mathbf{R}}^d_+$ on the neighborhood U of p. When $p \in \overline{\Sigma_2 \cup \Sigma_3} \setminus N$ we may assume, by Lemma 4, that in the coordinate system $(x_1, \ldots, x_d)$ the first order differential operator B is expressed as

$$B = \beta^d \frac{\partial}{\partial x_d}, \qquad \beta^d > 0, \tag{15}$$

on $\partial M \cap U = \{(x_1, \ldots, x_d) \in U: x_d = 0\}$.

PROPOSITION 1. *Assume that for a vector $v \in \mathbf{R}^d$ one of the following conditions is satisfied*:

(i) $p \in M \setminus \overline{\Sigma_2 \cup \Sigma_3}$, $\langle a(p)v, v\rangle > 0$,
(ii) $p \in \overline{\Sigma_2 \cup \Sigma_3} \setminus N$, $\langle a(p)v, v\rangle > 0$, $\langle e_d, v\rangle \geqslant 0$,
(iii) $p \in N \setminus \overline{\Gamma_2 \cup \Gamma_3}$, $\langle \beta(p), v\rangle > 0$,
(iv) $p \in \overline{\Gamma_2 \cup \Gamma_3}$, $\langle a(p)v, v\rangle > 0$, $\langle \beta(p), v\rangle > 0$.

Then there exists a sequence $\{p^n\}_{n=1}^{\infty}$ such that $p^n \to p$ as $n \to \infty$, $p^n \in M \cap B(p + v, |v|)$ and $u(p^n) = u(p)$ for $n = 1, 2, \ldots$.

Here $B(p + v, |v|)$ denotes the open ball of radius $|v|$ and center $p + v$ in $\mathbf{R}^d$, i.e., $B(p + v, |v|) = \{x \in \mathbf{R}^d: |x - (p + v)| < |v|\}$.

PROOF. Consider the function

$$f(x) = \exp\left(-C\left|x - \left(p + \tfrac{1}{2}v\right)\right|^2\right) - \exp\left(-C\left|\tfrac{1}{2}v\right|^2\right),$$

where C is a sufficiently large positive constant. Direct computation shows that $\langle a(p)v, v\rangle > 0$ and $\langle \beta(p), v\rangle > 0$ imply $Af(p) > 0$ and $Bf(p) > 0$ respectively. When $p \in \overline{\Sigma_2 \cup \Sigma_3} \setminus N$, since B is of the form (15), $\langle e_d, v\rangle \geqslant 0$ gives $Bf \geqslant 0$ on a neighborhood of p on M. Hence, by Lemma 6, there exists a sequence $\{p^n\}_{n=1}^{\infty}$ such that $p^n \to p$ as $n \to \infty$,

$$\begin{aligned} p^n &\in \{x \in M: f(x) \geqslant 0\} \setminus \{p\} \\ &= M \cap \bar{B}\left(p + \tfrac{1}{2}v, \left|\tfrac{1}{2}v\right|\right) \setminus \{p\} \subset M \cap B(p + v, |v|) \end{aligned}$$

and $u(p^n) = u(p)$ for $n = 1, 2, \ldots$; this completes the proof.

PROOF OF THEOREM 1 (*Step* 1). We set

$$F = \{x \in U: u(x) = u(p)\}$$

and take an $\mathfrak{X}_1$-propagation path $x(t)$, $t \geqslant 0$, with $x(0) = p$, arbitrarily. By Lemma 3, it will suffice to check the following condition for a sufficiently small positive constant T: For any $t \in [0, T]$

$$\langle X(y), x(t) - y\rangle \leqslant 0$$

whenever $y \in F$ and $|x(t) - y| = \operatorname{dist}(x(t), F)$, where X is an $\mathfrak{X}_1$-propagation vector field satisfying the conditions (6), (7) and (8). We shall deduce a contradiction from the assumption that for some $s \in [0, T]$ and some $q \in \{y \in F: |x(s) - y| = \operatorname{dist}(x(s), F)\}$ the inequality

$$\langle X(q), x(s) - q\rangle > 0 \tag{16}$$

holds. Here we note that if $X(q) \in \Delta_A(q)$, i.e., $X(q) = \sum_{i=0}^{d} \lambda_i X_i(q)$ for some $\lambda_0 \in \overline{\mathbf{R}}_+$, $\lambda_i \in \mathbf{R}$, $1 \leqslant i \leqslant d$, then the inequality (16) implies, by Lemma 1, either

$$-X(q) \notin \Delta_A(q), \qquad \langle X_0(q), x(s) - q\rangle > 0 \tag{17}$$

or

$$\langle a(q)(x(s) - q), x(s) - q \rangle > 0. \tag{18}$$

If $X(q) \in \Delta_B(q)$ then the inequality (16) gives

$$\langle \beta(p), x(s) - q \rangle > 0. \tag{19}$$

Clearly, when $q \in \partial M$,

$$\langle e_d, x(s) - q \rangle \geqslant 0. \tag{20}$$

We first consider the case that $-X \in \Delta_A$ on $(M \setminus N) \cup \overline{\Gamma_2 \cup \Gamma_3}$. Proposition 1 shows that the inequalities (18), (19) and (20) imply $B(x(s), \operatorname{dist}(x(s), F)) \cap F \neq \varnothing$; this is a contradiction. Hence $u \equiv u(p)$ on the $\mathfrak{X}_1$-propagation path $\{x(t): t \geqslant 0\}$ if $-X \in \Delta_A$ on $(M \setminus N) \cup \overline{\Gamma_2 \cup \Gamma_3}$.

PROPOSITION 2 ([1]). *Assume that the point $p \in M \setminus (\mathring{\Sigma}_2 \cup \overline{\Sigma}_3 \cup N)$ and the inequality*

$$\langle X_0(p), v \rangle > 0$$

holds for a vector $v \in \mathbf{R}^d$. Then there exists a sequence $\{p^n\}_{n=1}^{\infty}$ such that $p^n \to p$ as $n \to \infty$, $p^n \in M \cap B(p + v, |v|)$ and $u(p^n) = u(p)$ for $n = 1, 2, \ldots$.

Here $\mathring{\Sigma}_2$ denotes the interior of the set Σ_2 relative to the topology of ∂M. The above proposition is contained in Propositions 2 and 4 in [1].

PROOF OF THEOREM 1 (*Step* 2). In this step we consider the case that $-X \in \Delta_A$ on $((\mathring{\Sigma}_2 \cup \overline{\Sigma}_3) \setminus (\Sigma_2 \cup N)) \cup \overline{\Gamma_2 \cup \Gamma_3}$; by the condition (5), this is equivalent with $-X \in \Delta_A$ on $((\mathring{\Sigma}_2 \cup \overline{\Sigma}_3) \setminus N) \cup \overline{\Gamma_2 \cup \Gamma_3}$. Propositions 1 and 2 show that the inequalities (17), (18), (19) and (20) give a contradiction $B(x(s), \operatorname{dist}(x(s), F)) \cap F \neq \varnothing$. Thus $u \equiv u(p)$ on the $\mathfrak{X}_1$-propagation path $\{x(t): t \geqslant 0\}$ if $-X \in \Delta_A$ on $((\mathring{\Sigma}_2 \cup \overline{\Sigma}_3) \setminus (\Sigma_2 \cup N)) \cup \overline{\Gamma_2 \cup \Gamma_3}$.

PROPOSITION 3. *Assume that the point $p \in \partial M \setminus N$ and $x(t)$, $t \geqslant 0$, is an $\mathfrak{X}_1$-propagation path with $x(0) = p$. If there is a sequence $\{p^n\}_{n=1}^{\infty}$ such that $p^n \to p$ through $\mathring{M} = M \setminus \partial M$ as $n \to \infty$ and $u(p^n) = u(p)$ for $n = 1, 2, \ldots,$ then $u \equiv u(p)$ on $\{x(t): 0 \leqslant t \leqslant T\}$ for a sufficiently small positive constant T.*

PROOF. By X we denote the $\mathfrak{X}_1$-propagation vector field satisfying the conditions (6), (7) and (8). Since X is C^1-smooth on a neighborhood of p on M and X never points out of M on ∂M, we can show, by Lemma 3, that for some small $T > 0$ independent of $n = 1, 2, \ldots$ there exists a sequence $\{x(t, p^n), 0 \leqslant t \leqslant T\}_{n=1}^{\infty}$ of $\mathfrak{X}_1$-propagation paths satisfying

$$\{x(t, p^n): 0 \leqslant t \leqslant T\} \subset \mathring{M},$$
$$\dot{x}(t, p^n) = X(x(t, p^n)) \quad \text{on } [0, T], \qquad x(0, p^n) = p^n.$$

In view of the result proved in Step 2 of the proof of Theorem 1, $u \equiv u(p)$ on each $\mathfrak{X}_1$-propagation path $\{x(t, p^n): 0 \leqslant t \leqslant T\}$. Hence, letting $n \to \infty$, we obtain the desired result.

PROOF OF THEOREM 1 (*Step* 3). If the point $q \in \Sigma_3 \setminus N$ then $\langle X_d, e_d \rangle > 0$ at q. Clearly X_d is a C^1 vector field on the open set U with $\pm X_d \in \Delta_A$ on U. Thus the result proved in Step 1 insures the existence of a sequence $\{q^n\}_{n=1}^{\infty}$ satisfying

$q^n \to q$ through $\mathring{M}$ as $n \to \infty$ and $u(q^n) = u(q)$ for $n = 1, 2, \ldots$. From the inequality (16) and Proposition 3 we can deduce the desired contradiction $B(x(s), \operatorname{dist}(x(s), F)) \cap F \neq \varnothing$.

If $q = (q_1, \ldots, q_{d-1}, 0) \in (\overline{\Sigma}_3 \cap \Sigma_1) \setminus N$ then we consider the auxiliary function

$$f(x) = x_d - c \sum_{i=1}^{d-1} (x_i - q_i)^2,$$

where c is a sufficiently small positive constant. By performing a suitable coordinate transformation, keeping x_d fixed, we may assume that

$$a(q) = \begin{pmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{r} & 0 \\ 0 & \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} \end{pmatrix}$$

where $0 \leqslant r = \operatorname{rank} a(q) \leqslant d - 1$. Direct computation gives, by Lemma 1,

$$Af(q) \geqslant -2rc + \sum_{j=1}^{r} \frac{\partial a^{dj}}{\partial x_j}(q) + \langle X_0(q), e_d \rangle.$$

In case $\sum_{j=1}^{r} \partial a^{dj}(q)/\partial x_j \geqslant 0$, Lemma 6 shows that there is a sequence $\{q^n\}_{n=1}^{\infty}$ which tends to q through $\{x \in M: f(x) \geqslant 0\} \setminus \{q\} \subset \mathring{M}$ and $u(q^n) = u(q)$ for $n = 1, 2, \ldots$. Thus, by Proposition 3, the inequality (16) leads to the desired contradiction $B(x(s), \operatorname{dist}(x(s), F)) \cap F \neq \varnothing$. In case $\partial a^{dk}(q)/\partial x_k < 0$ for some $k = 1, \ldots, r$, we extend the C^∞ vector field X_k on $M \cap U = \overline{\mathbf{R}}^d_+ \cap U$ so as to be C^∞-smooth on a neighborhood of p in $\mathbf{R}^d$, and consider the integral curve $y(t) = (y_1(t), \ldots, y_d(t))$, $0 \leqslant t \leqslant T$, satisfying

$$\dot{y}(t) = -X_k(y(t)) \quad \text{on } [0, T], \qquad y(0) = q,$$

for a sufficiently small positive constant T. Since $y_d(0) = 0$, $\dot{y}_d(0) = -a^{dk}(q) = 0$ and $\ddot{y}_d(0) = -\partial a^{dk}(q)/\partial x_k > 0$, we obtain

$$y_d(t) = -\frac{1}{2} \frac{\partial a^{dk}}{\partial x_k}(q) t^2 + o(t^2)$$

as $t \downarrow 0$; this implies $\{y(t): 0 < t \leqslant T\} \subset \mathring{M}$. As in Step 1, using Lemma 3 and Proposition 1, we can show that $u \equiv u(q)$ on $\{y(t): 0 < t \leqslant T\} \subset \mathring{M}$; the condition in Proposition 3 is fulfilled. By Proposition 3, the inequality (16) gives the desired contradiction. Therefore $u \equiv u(p)$ on the $\mathfrak{X}_1$-propagation path $\{x(t): t \geqslant 0\}$ if $-X \in \Delta_A$ on $\{(\partial\Sigma_3 \cap \partial\Sigma_0) \setminus N\} \cup \overline{\Gamma_2 \cup \Gamma_3}$, where $\partial\Sigma_3$ and $\partial\Sigma_0$ are respectively the boundaries of the sets Σ_3 and Σ_0 relative to the topology of ∂M.

PROPOSITION 4. *Assume that the point $p \in (\partial\Sigma_3 \cap \partial\Sigma_0) \setminus N$ and a vector $v \in \mathbf{R}^d$ satisfies the inequality*

$$\langle X_0(p), v \rangle > 0. \tag{21}$$

Then there exists a sequence $\{p^n\}_{n=1}^{\infty}$ such that $p^n \to p$ as $n \to \infty$, $p^n \in M \cap B(p + v, |v|)$ and $u(p^n) = u(p)$ for $n = 1, 2, \ldots$.

PROOF. In view of Propositions 1 and 3 we have only to consider the case

$$\langle a(p)v, v\rangle = 0 \tag{22}$$

and

$$u < u(p) \quad \text{on } \mathring{M} \cap U. \tag{23}$$

Since $\langle X_0(p), e_d\rangle = 0$ and $a(p)e_d = 0$, the inequality (21) and the equation (22) remain true for the projected vector $\tilde{v} = (v_1, \ldots, v_{d-1}, 0)$ instead of the vector $v = (v_1, \ldots, v_{d-1}, v_d)$, i.e., $\langle X_0(p), \tilde{v}\rangle > 0$ and $\langle a(p)\tilde{v}, \tilde{v}\rangle = 0$. By performing a suitable rotation of coordinates, keeping x_d fixed, we may assume that $\tilde{v} = \rho e_1 = (\rho, 0, \ldots, 0)$ for some positive constant ρ and

$$a(p) = \begin{bmatrix} 0 & & & & & & \\ & & \overbrace{}^{r} & & & & 0 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & 0 & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

where $0 \leqslant r = \operatorname{rank} a(p) \leqslant d - 2$. Without loss of generality, we assume that p is the origin o in $\mathbf{R}^d$.

We first show $\partial a^{1d}(p)/\partial x_d = 0$. If $\partial a^{1d}(p)/\partial x_d \neq 0$ then $\partial a^{dd}(p)/\partial x_d > 0$, since for any $t > 0$

$$\frac{|a^{1d}(p + te_d)|^2}{t^2} \leqslant \frac{a^{11}(p + te_d)}{t}\,\frac{a^{dd}(p + te_d)}{t}.$$

Let us consider an auxiliary function

$$f(x) = x_d - c\sum_{i=1}^{d-1} x_i^2$$

where c is a sufficiently small positive constant. Direct computation easily gives

$$Af(p) = -2rc + \sum_{j=2}^{r+1} \frac{\partial a^{dj}}{\partial x_j}(p) + \frac{\partial a^{dd}}{\partial x_d}(p).$$

Hence, as in Step 3 of the proof of Theorem 1, we can show in either case $\sum_{j=2}^{r+1} \partial a^{ij}(p)/\partial x_j \geqslant 0$ or < 0 that there is a sequence of the points in $\{x \in M: u(x) = u(p)\}$ which tends to p through $\mathring{M}$. This contradicts the assumption (23).

Next we consider the auxiliary function

$$g(x) = x_1 - \frac{1}{2}\sum_{j=2}^{r+1} \frac{\partial a^{ij}}{\partial x_j}(p)x_j^2 - \sum_{\substack{i,j=2\\ i\neq j}}^{r+1} \frac{\partial a^{1i}}{\partial x_j}(p)x_i x_j$$
$$-2c\sum_{j=2}^{r+1} x_i^2 - 2C\left(x_1^2 + \sum_{i=r+2}^{d-1} x_i^2\right),$$

where c and C are sufficiently small and sufficiently large positive constants respectively. Since $f(p) = 0$,

$$Ag(p) = -4rc + \langle X_0(p), \tilde{v} \rangle > 0$$

and

$$Bg = \beta^d \frac{\partial f}{\partial x_d} \equiv 0,$$

Lemma 6 and (23) show that there is a sequence $\{q^n\}_{n=1}^{\infty}$ such that $u(q^n) = u(p)$ for $n = 1, 2, \ldots$ and $q^n \to p$ through $\{x \in \partial M: g(x) > 0\}$ as $n \to \infty$; this implies that

$$\begin{aligned} q_1^n > \frac{1}{2} \sum_{j=2}^{r+1} \frac{\partial a^{1j}}{\partial x_j}(p)(q_j^n)^2 + \sum_{\substack{i,j=2 \\ i \neq j}}^{r+1} \frac{\partial a^{1i}}{\partial x_j}(p) q_i^n q_j^n \\ + c \sum_{j=2}^{r+1} (q_j^n)^2 + C\left\{ (q_1^n)^2 + \sum_{i=r+2}^{d-1} (q_i^n)^2 \right\} \end{aligned} \tag{24}$$

for every $q^n = (q_1^n, \ldots, q_{d-1}^n, q_d^n) = (q_1^n, \ldots, q_{d-1}^n, 0)$. Let us consider integral curves $x^k(t, q^n) = (x_1^k(t, q^n), \ldots, x_d^k(t, q^n))$, $2 \leqslant k \leqslant r+1$, $1 \leqslant n < \infty$, defined as follows:

$$\begin{aligned} &\dot{x}^2 = \tilde{X}_2(x^2), \qquad x^2(0, q^n) = q^n, \\ &\dot{x}^3 = \tilde{X}_3(x^3), \qquad x^3(0, q^n) = x^2(-q_2^n, q^n), \\ &\quad\vdots \\ &\dot{x}^{r+1} = \tilde{X}_{r+1}(x^{r+1}), \\ &x^{r+1}(0, q^n) = x^r(-x_r^{r-1}(\ldots -x_4^3(-x_3^2(-q_2^n, q^n), q^n), \ldots, q^n), q^n), \end{aligned}$$

where $\tilde{X}_k = \sum_{j=1}^{d-1} a^{kj} \partial/\partial x_j$, $2 \leqslant l \leqslant r+1$. As in Step 1 of the proof of Theorem 1, using Lemma 3, Proposition 1 and the assumption (23), we can show that the function u identically equals $u(p)$ on each integral curve $x^k(t, q^n)$, $2 \leqslant k \leqslant r+1$, $1 \leqslant n < \infty$. For convenience sake we set

$$\begin{aligned} p^n &= (p_1^n, \ldots, p_d^n) \\ &= x^{r+1}(-x_{r+1}^r(-x_r^{r-1}(\ldots -x_4^3(-x_3^2(-q_2^n, q^n), q^n), \ldots, q^n), q^n)). \end{aligned}$$

By Lemma 2 and the inequality (24) we obtain

$$\sum_{i=2}^{d-1} (p_i^n)^2 \leqslant O(1) \sum_{j=2}^{r+1} (q_j^n)^4 + O(1)\left\{ (q_1^n)^2 + \sum_{i=r+2}^{d-1} (q_i^n)^2 \right\}$$

and

$$\begin{aligned} p_1^n &= q_1^n - \frac{1}{2} \sum_{j=2}^{r+1} \frac{\partial a^{1j}}{\partial x_j}(p)(q_j^n)^2 - \sum_{\substack{i,j=2 \\ i \neq j}}^{r+1} \frac{\partial a^{1i}}{\partial x_j}(p) q_i^n q_j^n + o(|q^n|^2) \\ &> (c + o(1)) \sum_{j=2}^{r+1} (q_j^n)^2 + (C + o(1))\left\{ (q_1^n)^2 + \sum_{i=r+2}^{d-1} (q_i^n)^2 \right\} \end{aligned}$$

as $n \to \infty$. Therefore, for all sufficiently large n, we have

$$\begin{aligned} p_1^n(2\rho - p_1^n) &\geqslant \rho p_1^n \\ &> \rho(c + o(1)) \sum_{j=2}^{r+1} (q_j^n)^2 + \rho(C + o(1)) \left\{ (q_1^n)^2 + \sum_{i=r+2}^{d-1} (q_i^n)^2 \right\} \\ &\geqslant O(1) \sum_{j=2}^{r+1} (q_j^n)^4 + O(1) \left\{ (q_1^n)^2 + \sum_{i=r+2}^{d-1} (q_i^n)^2 \right\} \\ &\geqslant \sum_{i=2}^{d-1} (p_i^n)^2, \end{aligned}$$

i.e., $p^n \in B(\rho e_1, \rho) = B(p + \tilde{v}, |\tilde{v}|) \subset B(p + v, |v|)$. The proof of Proposition 4 is now complete.

Proof of Theorem 1 (*Step* 4). Proposition 4 shows that $u \equiv u(p)$ on the $\mathfrak{X}_1$-propagation path $\{x(t): t \geqslant 0\}$ if $-X \in \Delta_A$ on $\overline{\Gamma_2 \cup \Gamma_3}$.

If the point $q \in \overline{\Gamma_2 \cup \Gamma_3}$ and $X(q) \notin T_q(\partial N)$ then, by choosing a suitable chart, we may assume that q is the origin O in $\mathbf{R}^d$, $M = \{(x_1, \ldots, x_d) \in \mathbf{R}^d: x_d > 0\}$ and $N = \{(x_1, \ldots, x_d) \in \mathbf{R}^d: x_1 \leqslant 0, x_d = 0\}$. Clearly $\langle X(q), e_1 \rangle \neq 0$. When $a(q)e_1 \neq 0$, Proposition 1 insures the existence of a sequence $\{q^n\}_{n=1}^{\infty}$ which tends to q through $M \setminus \partial N$ as $n \to \infty$ and $u(q^n) = u(q)$ for $n = 1, 2, \ldots$. We consider a sequence $\{x(t, q^n), 0 \leqslant t \leqslant T\}_{n=1}^{\infty}$ of $\mathfrak{X}_1$-propagation paths satisfying

$$\dot{x}(t, q^n) = X(x(t, q^n)) \quad \text{on } [0, T], \qquad x(0, q^n) = q^n.$$

If we take $T > 0$ sufficiently small, then we obtain, by (5) and $\langle X(q), e_1 \rangle \neq 0$,

$$\{x(t, q^n): 0 \leqslant t \leqslant T\} \subset M \setminus \partial N$$

for all $n = 1, 2, \ldots$. Since we have already proved in the former three steps that $u \equiv u(p)$ on $\{x(t, q^n): 0 \leqslant t \leqslant T\}$, $n = 1, 2, \ldots,$ we obtain the desired contradiction $B(x(s), \operatorname{dist}(x(s), F)) \cap F \neq \varnothing$.

When $a(q)e_1 = 0$ and $a(q)e_d \neq 0$, since $\langle X_0(q), e_1 \rangle > 0$ and $\langle a(q)e_d, e_d \rangle > 0$, we can choose a vector $v \in \{\lambda_1 e_1 + \lambda_d e_d \in \mathbf{R}^d: \lambda_1 > 0, \lambda_d > 0\}$ so that $\langle X_0(q), v \rangle > 0$ and $\langle a(q)v, v \rangle > 0$. According to Proposition 1, there is a sequence $\{q^n\}_{n=1}^{\infty}$ which tends to q through $M \setminus N$ as $n \to \infty$ and $u(q^n) = u(q)$ for $n = 1, 2, \ldots$; as in the above case, this gives the desired contradiction.

When $a(q)e_1 = 0$ and $a(q)e_d = 0$, by performing a suitable coordinate transformation, keeping x_1 and x_d fixed, we may assume that

$$a(q) = \begin{bmatrix} 0 & & & & \\ & \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{r} & & & 0 \\ & & 0 & & \\ & 0 & & \ddots & \\ & & & & 0 \end{bmatrix}$$

where $0 \leqslant r = \operatorname{rank} a(q) \leqslant d-2$. Consider the auxiliary function

$$f(x) = x_1 - c\sum_{i=2}^{r+1} x_i^2,$$

where c is a sufficiently small positive constant. We easily have $f(q) = 0$,

$$Af(q) \geqslant -2cr + \sum_{j=2}^{r+1} \frac{\partial a^{1j}}{\partial x_j}(q) + \langle X_0(q), e_1\rangle$$

and $Bf(q) > 0$. In case $\sum_{j=2}^{r+1} \partial a^{1j}(q)/\partial x_j \geqslant 0$, by Lemma 6, we have the desired contradiction. In case $\partial a^{1k}(q)/\partial x_k < 0$ for some $k = 1, \ldots, r$, as in Step 3 of the proof of Theorem 1, we arrive at the desired contradiction. The proof of Theorem 1 is now complete.

4. Proof of Theorem 2. Throughout this section u denotes a solution of Venttsel′'s value problem (10) in $C^2(M)$, and we assume that u attains the maximum at a point p on M. We fix a chart $(U, (x_1, \ldots, x_d))$, with $p \in U$, and identify M with $\overline{\mathbf{R}}^d_+$ on the neighborhood U of p. When $p \in \overline{\Sigma_2 \cup \Sigma_3} \setminus N$ we may assume, by Lemma 5, that in the coordinate system $(x_1, \ldots, x_d)$ Venttsel′'s differential operator B is expressed as

$$B = \sum_{i,j=1}^{d-1} \alpha^{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \beta^d\frac{\partial}{\partial x_d}, \qquad \beta^d > 0, \tag{25}$$

on $\partial M \cap U = \{(x_1, \ldots, x_d) \in U \colon x_d = 0\}$.

The basic idea of the proof of Theorem 2 is essentially the same as that of Theorem 1. Instead of Proposition 1 we use the following proposition.

Proposition 5. *Assume that for a vector $v \in \mathbf{R}^d$ one of the following conditions is satisfied*:

(i) $p \in M \setminus (\overline{\Sigma}_3 \cup N)$, $\langle a(p)v, v\rangle > 0$,
(ii) $p \in \overline{\Sigma}_3 \setminus N$, $\langle a(p)v, v\rangle > 0$, $\langle \alpha(p)v, v\rangle > 0$,
(iii) $p \in N \setminus \overline{\Gamma_2 \cup \Gamma_3}$, $\langle \alpha(p)v, v\rangle > 0$,
(iv) $p \in \overline{\Gamma_2 \cup \Gamma_3}$, $\langle a(p)v, v\rangle > 0$, $\langle \alpha(p)v, v\rangle > 0$.

Then there exists a sequence $\{p^n\}_{n=1}^\infty$ such that $p^n \to p$ as $n \to \infty$, $p^n \in M \cap B(p+v, |v|)$ and $u(p^n) = u(p)$ for $n = 1, 2, \ldots$.

Proof. In view of Lemma 6 and Proposition 1 we have only to check the case $p \in \overline{\Sigma}_2 \setminus (\overline{\Sigma}_3 \cup N)$. Let us consider a differential operator

$$\tilde{B} = \sum_{i,j=1}^{d-1} (\beta^d a^{ij} - b^d \alpha^{ij})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d-1} (\beta^d b^i - b^d \beta^i)\frac{\partial}{\partial x_i}. \tag{26}$$

It is easy to show $\tilde{B}u \geqslant 0$ on Σ_2 and $\langle(\beta^d(p)a^{ij}(p) - b^d(p)\alpha^{ij}(p))v, v\rangle > 0$. Hence, by applying Lemma 6 to the function

$$f(x) = \exp\left(-C\left|x - \left(p + \tfrac{1}{2}v\right)\right|^2\right) - \exp\left(-C\left|\tfrac{1}{2}v\right|^2\right),$$

where C is a sufficiently large positive constant, we obtain the desired fact.

We can show the same fact as Proposition 2 when B is of Venttsel′'s type. The proof is almost the same except we use the differential operator $\tilde{B}$ defined by (26) instead of B. Clearly, Proposition 3 remains true. However, we do not know whether Proposition 4 remains valid for the solutions of Venttsel′'s boundary value problem or not.

In the proof of Theorem 2 we use the auxiliary function

$$f(x) = \exp\left(-C|x - (p + Ce_d)|^2\right) - \exp\left(-C|Ce_d|^2\right)$$

if $p \in \Sigma_3 \setminus N$, where C is a sufficiently large positive constant. Applying Lemma 6 to the function $f(x)$, we obtain a sequence $\{p^n\}_{n=1}^{\infty}$ which tends to p through $\mathring{M}$ as $n \to \infty$ and $u(p^n) = u(p)$ for $n = 1, 2, \ldots$. The remaining parts of the proof of Theorem 2 are parallel with that of Theorem 1.

5. Remarks. In this section we first consider the case that

> N is a k-dimensional, $1 \leqslant k \leqslant d - 2$, regular C^∞ submanifold of ∂M, with boundary ∂N.

We will show that on some additional assumptions the same theorems as Theorems 1 and 2 are valid. We set

$$N_0 = \{x \in N: \Delta_B(x) \subset T_x(N)\}, \qquad N_1 = N \setminus N_0,$$

and assume that

> N_0 is an l-dimensional, $0 \leqslant l \leqslant k$, regular C^∞ submanifold of ∂N, with boundary ∂N_0.

We classify the boundary ∂N_0 into four parts:

$\Gamma_3 = \{x \in \partial N_0$: At least, one of B-diffusion vectors is transversal to ∂N_0 at $x\}$,

$\Gamma_2 = \{x \in \partial N_0 \setminus \Gamma_3$: At least, one of B-drift vectors points out of N_0 at $x\}$,

$\Gamma_1 = \{x \in \partial N_0 \setminus \Gamma_3$: At least, one of B-drift vectors points into N_0 at $x\}$,

$\Gamma_0 = \{x \in \partial N_0 \setminus \Gamma_3$: All B-drift vectors are tangential to ∂N_0 at $x\}$.

We define the set $\mathfrak{X}$ of all propagation vector fields on M as follows: $X \in \mathfrak{X}$ if and only if

$$\begin{aligned}&X \text{ never points out of } M \text{ on } \partial M,\ X \in \Delta_A \text{ on}\\ &(M \setminus N_0) \cup \overline{\Gamma_2 \cup \Gamma_3},\ X \in \Delta_B \text{ on } N.\end{aligned} \tag{27}$$

$\mathfrak{X}_1$ (resp. $\mathfrak{X}_2$) is a subset of $\mathfrak{X}$ which consists of all propagation vector fields X satisfying the following condition:

$$-X(x) \in \Delta_A(x) \quad \text{if } x \in N_1 \cup \overline{\Gamma_2 \cup \Gamma_3} \text{ and } X(x) \in T_x(N) \tag{28}$$

$$\left(\text{resp. } -X(x) \in \Delta_A(x) \cap \Delta_B(x) \text{ if } x \in \partial\Sigma_3 \cup N_1 \cup \overline{\Gamma_2 \cup \Gamma_3}\right). \tag{29}$$

As in §1, from $\mathfrak{X}_k$ we construct the propagation sets $S_k(p)$, $k = 1, 2$. For such propagation sets $S_k(p)$, $k = 1, 2$, the corresponding statements to Theorems 1 and 2 remain valid.

In case

$$N \text{ consists of discrete points on } \partial M$$

we have only to replace the conditions (27) and (29) by

$$X \in \Delta_A \quad \text{on } M \setminus N_0,$$
$$-X \in \Delta_A(x) \cap \Delta_B(x) \quad \text{if } x \in \partial\Sigma_3 \cup N_1$$

respectively. In this case the condition (28) is a trivial one, since $X(x) \in T_x(N)$ implies $X(x) = 0$.

6. Examples.

EXAMPLE 1. Let $M = \{x \in \mathbf{R}^2 : |x| \leqslant 1\}$,

$$A = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$$

and

$$B = \beta^1 \frac{\partial}{\partial \tau} + \beta^2 \frac{\partial}{\partial n}, \qquad |\beta^1| + |\beta^2| \neq 0, \qquad \beta^2 \geqslant 0,$$

where $\partial/\partial\tau$ is a nonvanishing smooth vector field on ∂M and n is the inner normal vector on ∂M. Then

$$S_1(p) = \begin{cases} M & \text{if either } p \in \mathring{M} \text{ or } p \in \partial M,\ \beta^2 \not\equiv 0, \\ \partial M & \text{if } p \in \partial M,\ \beta^2 \equiv 0. \end{cases}$$

EXAMPLE 2. Let $M = \{(x_1, x_2) \in \mathbf{R}^2 : x_2 \geqslant 0\}$,

$$A = \frac{\partial}{\partial x_1}\left(a^{11} \frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(a^{22} \frac{\partial}{\partial x_2}\right) + b \frac{\partial}{\partial x_1}$$

and $B = \partial/\partial x_d$, where a^{11}, a^{22}, b are real smooth functions on M and

$$a^{11}(x) \begin{cases} = 0 & \text{if } x = (-1, 0) \text{ or } (1, 0), \\ > 0 & \text{otherwise;} \end{cases}$$
$$a^{22}(x) \begin{cases} = 0 & \text{if } x = (x_1, 0),\ -1 \leqslant x_1 \leqslant 1, \\ > 0 & \text{otherwise.} \end{cases}$$

We note that $\Sigma_3 = \{(x_1, 0) \in \mathbf{R}^2 : x_1 \notin [-1, 1]\}$ and $\Sigma_0 = \{(x_1, 0) \in \mathbf{R}^2 : x_1 \in [-1, 1]\}$. Clearly

$$S_1(p) = M \quad \text{for } p \in M \setminus \Sigma_0.$$

In case $b(-1, 0) \geqslant 0$ and $b(1, 0) \leqslant 0$

$$S_1(p) = \begin{cases} \{p\} & \text{if } p \in \partial\Sigma_0,\ b(p) = 0, \\ \Sigma_0 & \text{if } p \in \partial\Sigma_0,\ b(p) \neq 0, \\ \Sigma_0 & \text{if } p \in \mathring{\Sigma}_0. \end{cases}$$

In case $b(-1, 0) \geqslant 0$ and $b(1, 0) > 0$

$$S_1(p) = \begin{cases} \{p\} & \text{if } p = (-1, 0),\ b(p) = 0, \\ M & \text{otherwise.} \end{cases}$$

In case $b(-1, 0) < 0$ and $b(1, 0) > 0$

$$S_1(p) = M \quad \text{for } p \in M.$$

EXAMPLE 3. Let $M = \{(x_1, x_2) \in \mathbf{R}^2: x_2 \geqslant 0\}$,

$$A = a^{11}\frac{\partial^2}{\partial x_1^2} + a^{22}\frac{\partial^2}{\partial x_2^2}$$

and

$$B = a^{11}\frac{\partial^2}{\partial x_1^2} + \sum_{i=1}^{2} \beta^i \frac{\partial}{\partial x_i},$$

where a^{ii}, β^i are real smooth functions on M and

$$a^{11}\begin{cases} \equiv 0 & \text{on } \{x_1 = 0\}, \\ > 0 & \text{otherwise,} \end{cases}$$

$$a^{22}\begin{cases} \equiv 0 & \text{on } \{x_1 = 0\}, \\ > 0 & \text{on } \mathring{M} \setminus \{x_1 = 0\}, \\ > 0 & \text{at } (-2, 0) \text{ and } (2, 0), \end{cases}$$

$$\beta^2\begin{cases} \equiv 0 & \text{on } \{-1 \leqslant x_1 \leqslant 1, x_2 = 0\}, \\ > 0 & \text{on } \partial M \setminus \{-1 \leqslant x_1 \leqslant 1, x_2 = 0\}. \end{cases}$$

It is easy to show $\Sigma_3 = \{x \in \partial M: a^{22}(x) > 0\}$, $\Sigma_2 = \{x \in \partial M \setminus \Sigma_3: \partial a^{22}(x)/\partial x_2 > 0\}$, $\Sigma_1 = \{x \in \partial M \setminus \Sigma_3: \partial a^{22}(x)/\partial x_2 < 0\}$ and $\Sigma_0 = \{x \in \partial M \setminus \Sigma_3: \partial a^{22}(x)/\partial x_2 = 0\}$. In case $\beta^1(0) = 0$

$$S_2(p) = \begin{cases} \{x \in M: x_1 \leqslant 0\} & \text{if } p \in \{x \in M: x_1 < 0\}, \\ \{p\} & \text{if } p \in \{x \in M: x_1 = 0\}, \\ \{x \in M: x_1 \geqslant 0\} & \text{if } p \in \{x \in M: x_1 > 0\}. \end{cases}$$

In case $\beta^1(0) > 0$

$$S_2(p) = \begin{cases} M & \text{if } p \in \{x \in M: x_1 < 0\}, \\ \{p\} & \text{if } p \in \{x \in M: x_1 = 0\} \setminus \{0\}, \\ \{x \in M: x_1 \geqslant 0\} & \text{if } p \in \{x \in M: x_1 > 0\} \cup \{0\}. \end{cases}$$

EXAMPLE 4. Let $M = \{(x_1, x_2) \in \mathbf{R}^2: x_2 \geqslant 0\}$.

$$A = \frac{\partial^2}{\partial x_2^2}, \qquad B = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}.$$

Assume that a function $u \in C^2(M)$ satisfies the inequalities

$$Au \geqslant 0 \quad \text{on } M, \qquad Bu \geqslant 0 \quad \text{on } \Sigma_3 = \partial M$$

and that u attains the maximum at a point $p = (p_1, p_2)$ on M. Then Theorem 1 shows that

$$u \equiv u(p) \quad \text{on } S_1(p) = \{x \in M: x_1 = p_1\}.$$

In this example we obtain a more fine result

$$u \equiv u(p) \quad \text{on } \{x \in M: x_1 \geqslant p_1\}.$$

In fact, if we consider the auxiliary function

$$f(x) = \left\{\tfrac{1}{2}(x_1 - p_1) + x_2\right\}^2 + 2(x_1 - p_1) - x_2$$

then we obtain $f(p_1, 0) = 0$, $Af > 0$ and $Bf > 0$. Hence, by Lemma 6 and Theorem 1, there is a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$u \equiv u(p) \quad \text{on } \{x \in M: x_1 = p_1 + \varepsilon_n\}$$

for $n = 1, 2, \ldots$. Iteration of this argument gives the desired result.

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